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Collective radiation by harmonic oscillators

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Abstract. The radiation of a small system of harmonic oscillators is analysed. The exact solution of the problem in the dipole and rotating wave approximation is discussed. It is shown that only vibrations of the centre of charge are radiatively damped and that their damping constant is N times larger than for a single oscillator. Due to the collective emission, broadening and shift of the line, dependent on the number of oscillators, also occur.

1. Introduction

We consider the problem of the spontaneous emission of electromagnetic radiation by the system composed of N harmonic oscillators, oscillating with the frequency ω_0 and occupying a volume of small dimensions in comparison with the wavelength. Our approach is not based on perturbation theory. Simplifications of the hamiltonian lead to a solvable model. In the model a wide spectrum of radiation is taken into account. This allows for a direct description of the irreversible emission of radiation by the system.

A similar model has been considered by many authors. Louisell (1964) used it for the description of the mode damping inside resonant cavities. It was discussed by Schwabl and Thirring (1964) as a model of the laser. Some aspects of the emission of radiation by such a system were discussed by Agarwal (1971). In this paper we want to give a more complete discussion of the collective features of the emission process which appear in the damping of the source and in the spectral and time–space shape of the radiation. It is also shown that in the frequently used approximation of neglecting the A^2 interaction there are possible vibrations of the centre of charge of the system surrounded by its own radiation field, which do not decay. This effect disappears when the A^2 term is taken into account.

The model considered seems to be somewhat unrealistic. However, we hope that it allows for a better understanding of the radiation process. The model can be easily adapted to describe the cyclotron radiation emitted by a system of non-relativistic electrons moving in a magnetic field.

2. Specification of the hamiltonian

The hamiltonian of the N harmonic oscillators coupled to the electromagnetic field has the form:

$$H = \frac{1}{2m}\sum_{i=1}^{N}\left(\boldsymbol{p}_i - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2 + \frac{1}{2}m\omega_0^2\sum_{i=1}^{N}\boldsymbol{x}_i^2 + \frac{1}{8\pi}\int \mathrm{d}V(\boldsymbol{E}^2+\boldsymbol{B}^2), \tag{1}$$

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where $\boldsymbol{x}_i$ denotes the displacement of the ith charge from its equilibrium position, which is assumed to be fixed. The elastic force acting on each charge is treated as though external. We neglect the electrostatic interaction between the oscillators. The vector potential of the electromagnetic field is represented by a plane wave expansion with a continuous spectrum of the wavevectors and with a transverse field only:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2\pi}\sqrt{\hbar c}\sum_{\mu=1}^{2}\int \mathrm{d}_3k\frac{\boldsymbol{e}_{\boldsymbol{k}\mu}}{\sqrt{k}}(a_{\boldsymbol{k}\mu}\,\mathrm{e}^{\mathrm{i}\boldsymbol{kr}}+a^{\dagger}_{\boldsymbol{k}\mu}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kr}}), \tag{2}$$

where $\boldsymbol{e}_{\boldsymbol{k}\mu}$ are polarization vectors.

The creation and annihilation operators $a^{\dagger}_{\boldsymbol{k}\mu}$ and $a_{\boldsymbol{k}\mu}$ satisfy the commutation relations:

$$[a_{\boldsymbol{k}\mu}, a^{\dagger}_{\boldsymbol{p}\nu}] = \delta_{\mu\nu}\,\delta_3(\boldsymbol{k}-\boldsymbol{p}), \qquad [a_{\boldsymbol{k}\mu}, a_{\boldsymbol{p}\nu}] = 0 = [a^{\dagger}_{\boldsymbol{k}\mu}, a^{\dagger}_{\boldsymbol{p}\nu}]. \tag{3}$$

It will be convenient to describe also the oscillators by means of the creation and annihilation operators of their excitations:

$$\boldsymbol{B}_j = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega_0}\boldsymbol{x}_j+\mathrm{i}\frac{\boldsymbol{p}_j}{\sqrt{m\omega_0}}\right), \qquad \boldsymbol{B}^{\dagger}_j = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega_0}\boldsymbol{x}_j-\mathrm{i}\frac{\boldsymbol{p}_j}{\sqrt{m\omega_0}}\right). \tag{4}$$

We assume that the dimensions of the system are small in comparison with the wavelength of the emitted radiation $\lambda_{\mathrm{r}} \simeq 2\pi c/\omega_0$. Therefore, the interaction is taken into account only in the dipole approximation, ie we put $\mathrm{e}^{\mathrm{i}\boldsymbol{kr}} = 1$ in terms giving the coupling between the charges and the radiation. In the first part we have neglected terms coming from $\boldsymbol{A}^2$ in the interaction. Corrections caused by these terms are discussed in § 6. We also make the rotating wave approximation, ie we neglect terms $\boldsymbol{B}^{\dagger}_i a^{\dagger}_{\boldsymbol{k}\mu}$ and $\boldsymbol{B}_i a_{\boldsymbol{k}\mu}$ in the interaction. This is justified when we are only interested in the spectrum of the field near the resonant frequency.

With these approximations the hamiltonian (1) takes the form ($\omega = kc$):

$$H = \hbar\omega_0\sum_{j=1}^{N}\boldsymbol{B}^{\dagger}_j\boldsymbol{B}_j+\hbar\sum_{\mu}\int \mathrm{d}_3k\,\omega a^{\dagger}_{\boldsymbol{k}\mu}a_{\boldsymbol{k}\mu}$$
$$+\mathrm{i}\frac{e\hbar}{2\pi}\left(\frac{\omega_0}{2mc}\right)^{1/2}\sum_{\mu}\int \mathrm{d}_3k\,\boldsymbol{e}_{\boldsymbol{k}\mu}g(k)\left(a^{\dagger}_{\boldsymbol{k}\mu}\sum_{j=1}^{N}\boldsymbol{B}_j-a_{\boldsymbol{k}\mu}\sum_{j=1}^{N}\boldsymbol{B}^{\dagger}_j\right). \tag{5}$$

The form factor $g(k)$ should be equal to $k^{-1/2}$, but in that case the model becomes divergent. This divergence has an unphysical meaning and is connected with the invalidity of our dipole approximation for very large vectors $|\boldsymbol{k}|$ appearing because of the k integration. It can be removed by a renormalization procedure or by an appropriate cut-off of the integral. Therefore, we have introduced a form factor $g(k)$ vanishing sufficiently fast for large $|\boldsymbol{k}|$. Sometimes we use a function $g(k)$ with a sharp cut-off at $k = k_{\max}$, ie

$$g(k) = \begin{cases} k^{-1/2} & k \leqslant k_{\max} \\ 0 & k > k_{\max}. \end{cases} \tag{6}$$

The natural parameter for the cut-off is $k_{\max} = 2\pi/d_0$, where d_0 denotes the dimension of the radiative system.

Because of the form of the hamiltonian (5) it is convenient as in Rzażewski and Żakowicz (1971) and Katriel and Adams (1970), to change the variables describing the system of oscillators. We perform a linear transformation of variables:

$$\boldsymbol{b}_k = \sum_{i=1}^{N} u_{ki}\boldsymbol{B}_i, \tag{7}$$

where u_{ik} is a unitary matrix such that

$$\boldsymbol{b}_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \boldsymbol{B}_i \equiv \boldsymbol{b}. \tag{8}$$

The hamiltonian of the system in the new variables is obtained in the form:

$$H = H_{\rm u} + H_{\rm c}, \tag{9}$$

where

$$H_{\rm u} = \hbar\omega_0 \sum_{i=2}^{N} \boldsymbol{b}_i^{\dagger}\boldsymbol{b}_i, \tag{10}$$

$$H_{\rm c} = \omega_0\hbar\boldsymbol{b}^{\dagger}\boldsymbol{b} + \hbar \int \mathrm{d}_3k\omega a_{\boldsymbol{k}\mu}^{\dagger}a_{\boldsymbol{k}\mu} + \mathrm{i}\hbar\lambda \sum_{\mu} \int \mathrm{d}_3k\boldsymbol{e}_{\boldsymbol{k}\mu}g(k)(\boldsymbol{b}a_{\boldsymbol{k}\mu}^{\dagger} - \boldsymbol{b}^{\dagger}a_{\boldsymbol{k}\mu}) \tag{11}$$

and

$$\lambda = \frac{e}{2\pi}\left(\frac{\omega_0 N}{2mc}\right)^{1/2}.$$

The hamiltonian $H_{\rm u}$ corresponds to those degrees of freedom which are not coupled to the field. The excitation of that part, within our approximation, does not influence the radiation of the system. It causes free and undamped oscillations with the frequency ω_0. The system is coupled to the field only via the first new variable, ie $\boldsymbol{b}$, which corresponds to the vibration of the centre of charge (or mass), which is denoted in the following as CCV. This part of the system coupled with the radiation has been separated to form the hamiltonian $H_{\rm c}$. In this way we have reduced the investigation of the radiation by the composite system to the analysis of the radiation by one oscillator. The composite character of the system under consideration is reflected only in the dependence of the coupling constant λ on the number of oscillators, ie $\lambda \sim N^{1/2}$.

The case when the whole excitation energy of the system excites only CCV, corresponds to the super-radiant state of the system (Rzążewski and Żakowicz 1971). When also non-radiative degrees of freedom are excited, one usually speaks of a system with trapped radiation.

Now we are going to solve the radiation emission problem, solving the Heisenberg equation of motion for the operators $a_{\boldsymbol{k}\mu}^{\dagger}$ and $\boldsymbol{b}^{\dagger}$.

3. Heisenberg equations for fields and oscillators

The hamiltonian (9) leads to the following equations of motion, in the Heisenberg picture, for the operators $\boldsymbol{b}^{\dagger}$ and $a_{\boldsymbol{k}\mu}^{\dagger}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{b}^{\dagger} = \mathrm{i}\omega_0\boldsymbol{b}^{\dagger} - \lambda \sum_{\mu} \int \mathrm{d}_3k\boldsymbol{e}_{\boldsymbol{k}\mu}g(k)a_{\boldsymbol{k}\mu}^{\dagger} \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a_{\boldsymbol{k}\mu}^{\dagger} = \mathrm{i}\omega a_{\boldsymbol{k}\mu}^{\dagger} + \lambda g(k)\boldsymbol{e}_{\boldsymbol{k}\mu}\boldsymbol{b}^{\dagger}. \tag{13}$$

The solution of these equations, which can be found by the method of Laplace transformation, will be written in the following form:

$$b^{\dagger}(t) = F(t)b^{\dagger}(0) - \lambda \sum_{\mu} \int \mathrm{d}_3 k e_{k\mu} g(k) G(k, t) a^{\dagger}_{k\mu}(0) \tag{14}$$

$$a^{\dagger}_{k\mu}(t) = a^{\dagger}_{k\mu}(0)\,\mathrm{e}^{\mathrm{i}\omega t} + \lambda g(k) G(k, t) e_{k\mu} b^{\dagger}(0) - \lambda^2 g(k) e_{k\mu} \sum_{\nu} \int \mathrm{d}_3 p e_{p\nu} g(p) J(k, p, t) a^{\dagger}_{p\nu}(0), \tag{15}$$

where the functions $F(t)$, $G(k, t)$ and $J(k, p, t)$ are given by the inverse Laplace integrals,

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\mathrm{e}^{zt}}{h(z)} \mathrm{d}z, \tag{16}$$

$$G(k, t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\mathrm{e}^{zt}}{h(z)(z - \mathrm{i}ck)} \mathrm{d}z, \tag{17}$$

$$J(k, p, t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\mathrm{e}^{zt}}{h(z)(z - \mathrm{i}ck)(z - \mathrm{i}pc)} \mathrm{d}z, \tag{18}$$

and

$$h(z) = z - \mathrm{i}\omega_0 + \frac{8\pi}{3}\lambda^2 \int_0^{\infty} \mathrm{d}k \frac{k^2 (g(k))^2}{z - ikc}. \tag{19}$$

The contour Γ is parallel to the imaginary axis and lies to the right of all singularities of the integrands. As will be seen later, this contour can be chosen to the right of the imaginary axis. Sometimes we shall use the function $h(z)$ corresponding to the form factor $g(k)$ given by (6), in which case

$$h(z) = z - \mathrm{i}\left(\omega_0 - \frac{8\pi}{3}\frac{\lambda^2}{c} k_{\max}\right) + \frac{8\pi}{3}\frac{\lambda^2}{c^2} z \ln\left(\frac{z - \mathrm{i}ck_{\max}}{z}\right). \tag{20}$$

In the Heisenberg picture, one can give the following interpretation of the structure of the solutions (14) and (15). The first term of (14) corresponds to the damping of the initial CCV excitation due to the emission of radiation by the system. The second term describes a variation of the CCV excitation in the case when, initially, the electromagnetic field is excited. For the field operators given by (15), the first term contributes to a free propagation of the initial field, the second term gives the field emitted by the excited system, and the last is connected with the interaction of photons in the presence of sources. In the ordinary approach, this term corresponds to the scattered field.

In order to give a physical discussion of the behaviour of the oscillators and radiation field, we have to find the functions $F(t)$, $G(k, t)$ and $J(k, p, t)$ in a more applicable form. These functions can be evaluated by the theorem of residues. However, one needs to be a little careful because the function $h(z)$, which appears in the integrals, is multivalued with branch points at $z = 0$ and $z = \infty$ (or $z = \mathrm{i}k_{\max}c$ if one uses (6)). Therefore, not only the poles but also the contour integrals along the cut contribute to $F(t)$, $G(k, t)$ and $J(k, p, t)$. The most convenient choice of the cut is along the negative part of the real axis (figure 1). The function $h(z)$ in the second quadrant is given by the analytic continuation of that in the first quadrant. If we use the function $h(z)$ given by (20), we should also introduce a second cut between the points $\mathrm{i}k_{\max}c$ and $-\infty$. However, when $\omega_0 \ll k_{\max}c$, which is the case, this second cut causes very small corrections and can therefore be neglected. In the

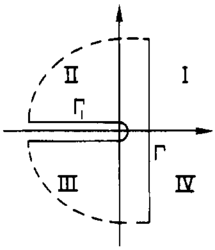

Figure 1. Contour of integration with cut along negative part of the real axis.

z plane, with a cut as described above, one can easily prove that zeros of $h(z)$ can only lie either in the second quadrant or on the negative part of the imaginary axis. Only zeros lying in the second quadrant have a physical meaning. However, it is also interesting to find the second type of zeros and to recognize their consequences and origins.

On the negative part of the imaginary axis $\mathrm{Re}\, h(\mathrm{i}y) = 0$, and the zeros of $h(z)$ on this axis are determined by the solutions of the equation $\mathrm{Im}\, h(\mathrm{i}y) = 0$ which, written explicitly, has the form

$$y = \omega_0 + \frac{8\pi}{3}\lambda^2 \int_0^\infty \mathrm{d}k \frac{k^2(g(k))^2}{y - kc}, \qquad y < 0. \tag{21}$$

This equation has a unique solution if

$$\lambda^2 \geqslant \frac{3}{8\pi} c\omega_0 d, \tag{22}$$

where

$$d = \left(\int_0^\infty \mathrm{d}k k (g(k))^2\right)^{-1}.$$

This condition can be satisfied if the number of oscillators is sufficiently large. As can be seen from our further discussion, the zero of $h(z)$ lying on the imaginary axis would correspond to the incomplete damping of initially excited CCV. In other words, there would be possible bound states of the system with excited CCV surrounded by an excited radiation field. However, when the condition for this effect is fulfilled, it is not justifiable to neglect the A^2 term in the interaction. This term may only be neglected for systems which are not coupled too strongly with the radiation and then such bound states do not appear.

Remaining in our approximation, ie using the hamiltonian (9), we restrict further discussion to the case of weak coupling, $\lambda \ll c$, for which the bound states mentioned above do not appear. Asymptotically, for $t \to \infty$, one must take into account contributions to the functions $F(t)$, $G(k, t)$ and $J(k, p, t)$ arising from the pole of $(h(z))^{-1}$ nearest to the imaginary axis. Treating λ/c as a small parameter, one finds that the solution, $z = -\gamma + \mathrm{i}\Omega$, of the equation $h(z) = 0$, when $h(z)$ is given by (20), is approximately given by

$$\Omega \simeq \omega_0 - \frac{8\pi}{3}\frac{\lambda^2}{c^2}\left(\omega_{\max} - \omega_0 \ln \frac{\omega_{\max}}{\omega_0}\right) \simeq \omega_0\left(1 - \frac{e^2 N \omega_{\max}}{3\pi m c^3}\right), \tag{23}$$

$$\gamma_0 \simeq \frac{8\pi^2\lambda^2}{3c^2}\Omega \simeq \frac{1}{3}\frac{e^2\omega_0^2}{mc^3}N. \tag{24}$$

We can now evaluate and express the integral (16) in the following form:

$$F(t) = \alpha \exp(-\gamma t + \mathrm{i}\Omega t) + C(t), \tag{25}$$

where

$$\alpha = \operatorname{res} z(h(z))^{-1}|_{z=-\gamma+\mathrm{i}\Omega} \simeq 1$$

and

$$C(t) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{0} \mathrm{d}x\, \mathrm{e}^{xt}\left(\frac{1}{h(x-\mathrm{i}0)} - \frac{1}{h(x+\mathrm{i}0)}\right). \tag{26}$$

Using an expansion of $h(x \pm \mathrm{i}0)$ for small values of $x < 0$, one can estimate the function $C(t)$ for large values of t, obtaining

$$C(t) \simeq \frac{\gamma}{\pi\omega_0^3 t^2}. \tag{27}$$

We see that $F(t) \to 0$ when $t \to \infty$, which corresponds to a complete damping of the CCV excitation.

One can also easily find the functions $G(k, t)$ and $J(k, p, t)$. For these functions, the terms due to the poles and cuts of $(h(z))^{-1}$ are similar to those which appear in the function $F(t)$. They describe the excitation of the system and the form of the field at a time just after the initial moment. These terms vanish in the limit $t \to \infty$. In addition to these, there are also terms coming from the poles lying on the imaginary axis. These terms are preserved in the limit $t \to \infty$ and describe the finally emitted and scattered radiation field of the system. They are also responsible for the conservation of the equal-time commutators for $t \to \infty$, eg $[\boldsymbol{b}(t), \boldsymbol{b}^{\dagger}(t)] = \hat{I}$, which can be checked by using the integral given in the appendix. Being most interested in the final field emitted by the system, we give only the explicit formula for the corresponding term:

$$G(k, t) \underset{t\to\infty}{=} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{h(\mathrm{i}\omega)}. \tag{28}$$

4. Radiative damping

In the present paper we are interested in the spontaneous radiation processes so we assume that the initial state of the system corresponds to the excited oscillators and the field in the vacuum state, ie

$$|\phi\rangle = |\psi_0\rangle|\Omega_{\mathrm{ph}}\rangle. \tag{29}$$

For the state $|\phi\rangle$ we shall now discuss the damping of the CCV which, as we have mentioned already, is the only degree of freedom radiatively damped. The dependence of the CCV excitation on time can be described by:

$$E_{\mathrm{CCV}}(t) = \hbar\omega_0\langle\phi \boldsymbol{b}^{\dagger}(t)\boldsymbol{b}(t)\phi\rangle, \tag{30}$$

which can be written:

$$E_{\mathrm{CCV}}(t) = E_{\mathrm{CCV}}(0)|F(t)|^2 \simeq E_{\mathrm{CCV}}(0)\left|\exp(-\gamma t + \mathrm{i}\Omega t) + \frac{\gamma}{\pi\omega_0^3 t^2}\right|^2, \tag{31}$$

where, cf (24),

$$\gamma = \frac{1}{3}\frac{e^2\omega_0^2}{mc^3}N.$$

We see that, in addition to the exponential damping of the CCV excitation, there is also a term describing non-exponential damping. This term dominates in the later stage of the evolution of the system. It becomes greater than the exponential term when

$$t \gg \frac{3}{\gamma}\ln\frac{\omega_0}{\gamma} = \frac{3}{\gamma}\ln\left(\frac{e^2N}{mc^3}\frac{1}{\omega_0}\right). \tag{32}$$

Because $\omega_0 \gg \gamma$, this happens after several relaxation periods and then the probability of CCV being in the ground state is almost unity, which means that the effect of non-exponential damping is very weak; usually it is neglected. There are also some attempts to avoid this non-exponential damping by referring to certain types of measurement procedures (cf, for example, Fonda *et al* 1973). However, in our opinion, one cannot rule out the possibility of non-exponential decay in general. For experimentally interesting times, the damping is described by the exponential term. The formula for this part of the damping of CCV was given in Agarwal (1971) and in a classical treatment in Fajn and Khanin (1969). It is worthwhile to point out that the damping parameter γ is N times larger than for a single oscillator and, therefore, the system loses its radiative energy faster than an isolated oscillator. This effect is due to the collective character of the emission.

5. Spectral and space–time pattern of radiation

Now we shall come to the discussion of the properties of the emitted radiation, all of which can be expressed by the operators $a^{\dagger}_{k\mu}(t)$ given by (15). We are interested here in the asymptotic form of the field. In the limit $t \to \infty$ we can drop the damping part of the expression (15) (both exponential and non-exponential). Of the remaining parts we are interested only in that which corresponds to the emitted radiation. Using (28) it can be written in the form:

$$a^{\dagger\mathrm{rad}}_{\boldsymbol{k}\mu}(t) = \lambda\frac{g(k)}{h(\mathrm{i}\omega)}\mathrm{e}^{\mathrm{i}\omega t}(\boldsymbol{e}_{\boldsymbol{k}\mu}\boldsymbol{b}^{\dagger}(0)). \tag{33}$$

The spectral and angular distributions of the emitted radiation, which can be deduced from the mean value of the photon number density operator corresponding to the photons of a given wavevector $\boldsymbol{k}$ and polarization μ,

$$N_{\boldsymbol{k}\mu}(t) = a^{\dagger\mathrm{rad}}_{\boldsymbol{k}\mu}(t)a^{\mathrm{rad}}_{\boldsymbol{k}\mu}(t), \tag{34}$$

is given by

$$S(\boldsymbol{k},\mu,t) = \langle\phi N_{\boldsymbol{k}\mu}(t)\phi\rangle = \lambda^2\frac{(g(k))^2}{|h(\mathrm{i}\omega)|^2}\langle\psi_0(\boldsymbol{e}_{\boldsymbol{k}\mu}\boldsymbol{b}^{\dagger}(0))(\boldsymbol{e}_{\boldsymbol{k}\mu}\boldsymbol{b}(0))\psi_0\rangle. \tag{35}$$

The function $S(\boldsymbol{k},\mu,t)$ can be written in the form:

$$S(\boldsymbol{k},\mu,t) = s(\omega)B_{\psi_0}(\hat{\boldsymbol{k}}), \tag{36}$$

where $B_{\psi_0}(\hat{\boldsymbol{k}})$ describes the angular distribution of photons and also their flux intensity, while $s(\omega)$ gives the pure spectral distribution of the radiation. The function $s(\omega)$ is chosen in the form

$$s(\omega) = \frac{8\pi}{3}\frac{\lambda^2}{c}\frac{k^2(g(k))^2}{|h(\mathrm{i}\omega)|^2}. \tag{37}$$

This function is normalized, $\int_0^\infty \mathrm{d}\omega s(\omega) = 1$, as can be shown using the integral given in the appendix. When the form factor $g(k)$ is given by (6), which corresponds to the function $h(z)$ being given by (20), the spectral distribution of the radiation is given by the formula ($0 \leqslant \omega \leqslant \omega_{\max}$):

$$s(\omega) = \frac{\frac{8}{3}\pi(\lambda^2/c^2)\omega}{\{\omega[1+\frac{8}{3}\pi\ln(\omega_{\max}-\omega)/\omega]-\omega_0+\frac{8}{3}\pi(\lambda^2/c^2)\omega_{\max}\}^2+[\frac{8}{3}\pi^2(\lambda^2/c^2)\omega]^2}. \tag{38}$$

For small values of the coupling constant, $\lambda \ll c$, this function has approximately a lorentzian shape:

$$s(\omega) = \frac{1}{3\pi}\frac{e^2\omega_0^2}{mc^3}N\left\{\left[\omega-\omega_0\left(1-\frac{e^2N\omega_{\max}}{3\pi mc^3}\right)\right]^2+\left(\frac{1}{3}\frac{e^2\omega_0^2}{mc^3}N\right)^2\right\}^{-1}. \tag{39}$$

We point out that the width of the spectral line and the position of its maximum depends on the number of radiating oscillators N. The shift of the central frequency and the line width are proportional to this number. Notice also that these effects are independent of the form of the initial excitation. Both effects exhibit, in the best way, the cooperative features of the emitted radiation.

Our spectral distribution was related to the spectral density of the number of photons. Multiplying it by $\hbar\omega$, one gets the spectral distribution for the energy.

Using the integral given in the appendix, one can show that the energy of the emitted photons is exactly equal to the energy of the initial excitation of the system, ie

$$E^{\mathrm{rad}}(t) = \hbar\sum_\mu\int \mathrm{d}_3k\omega a_{k\mu}^{\dagger\mathrm{rad}}(t)a_{k\mu}^{\mathrm{rad}}(t) = \hbar\omega_0\boldsymbol{b}^\dagger(0)\boldsymbol{b}(0). \tag{40}$$

Now we come to the description of the space–time properties of the emitted radiation. Putting $a_{k\mu}^{\dagger\mathrm{rad}}(t)$, which is given by (33), into (2), one could get the vector potential of the emitted radiation as a function of the space and time variables. However, some caution is needed. When deriving $a_{k\mu}^{\dagger\mathrm{rad}}(t)$ we introduced the form factor $g(k)$ instead of $k^{-1/2}$ to avoid divergences. We have previously neglected the photons of very high frequency and so we cannot use our formula for such photons now. The approximate vector potential of the radiation field is given by:

$$\boldsymbol{A}^{\mathrm{rad}}(\boldsymbol{r},t) = \frac{1}{2\pi}\sqrt{(\hbar c)}\sum_\mu\int \mathrm{d}_3k\,\boldsymbol{e}_{k\mu}g(k)(a_{k\mu}^{\dagger\mathrm{rad}}(t)\,e^{\mathrm{i}\boldsymbol{kr}}+a_{k\mu}^{\mathrm{rad}}\,e^{-\mathrm{i}\boldsymbol{kr}}). \tag{41}$$

Introducing $a_{k\mu}^{\dagger\mathrm{rad}}(t)$, given by (33), into this expression and performing the integration over the angular part, one finds $\boldsymbol{A}^{\mathrm{rad}}$ in the wave zone ($r \gg \lambda_r$),

$$\boldsymbol{A}^{\mathrm{rad}}(\boldsymbol{r},t) = \frac{2\lambda\sqrt{\hbar c}}{r}(\hat{I}-\boldsymbol{\hat{n}\hat{n}})\int_0^\infty \mathrm{d}kk(g(k))^2\sin(kr)\left(\frac{e^{\mathrm{i}\omega t}}{h(\mathrm{i}\omega)}\boldsymbol{b}^\dagger(0)+\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{h^*(\mathrm{i}\omega)}\boldsymbol{b}(0)\right). \tag{42}$$

Because of the function $\sin(kr)$, this expression represents incoming and outgoing spherical waves. Asymptotically, for large r and t, the incoming wave does not contribute to the integral, and A^{rad} can be represented in the form

$$A^{\text{rad}}(\boldsymbol{r}, t) = \mathrm{i}\frac{e}{2\pi}\left(\frac{\omega_0 \hbar}{2m}N\right)^{1/2}\frac{1}{r}(\hat{\boldsymbol{I}} - \widehat{\boldsymbol{nn}})(f(r-ct)\boldsymbol{b}^{\dagger}(0) - f^{*}(r-ct)\boldsymbol{b}(0)), \tag{43}$$

where

$$f(x) = \int_0^{\infty} \mathrm{d}k \frac{k(g(k))^2}{h(\mathrm{i}\omega)} \mathrm{e}^{-\mathrm{i}xk}. \tag{44}$$

This integral can be understood as the contour integral along the positive part of the real axis (contour Γ' in figure 2) of the complex k plane.

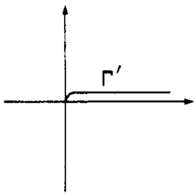

Figure 2. Contour of integration Γ'.

To estimate the integral (44) it is convenient to deform the contour in such a way that the factor $\exp(-\mathrm{i}kx)$ in the integrand becomes damped. Thus, if $x > 0$ the contour Γ' is replaced by Γ'' (figure 3). The integral along the arc γ'' tends to zero when its radius goes to infinity and the function $h(\mathrm{i}kc)$ has no zeros in the IV quadrant of the k plane. For $x < 0$ one changes the contour into Γ''' (figure 4), but in this case it is necessary to take into account contributions coming from the poles z_j of $h(\mathrm{i}kc)^{-1}$ lying in the I quadrant. One finds that

$$f(x) = -\int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{-xy}\frac{y(g(-\mathrm{i}y))^2}{h(yc)} \qquad x > 0 \tag{45}$$

$$f(x) = \frac{1}{2\pi\mathrm{i}}\sum_j \operatorname{res}\left(\frac{z(g(\mathrm{i}z))^2\, \mathrm{e}^{-zx}}{h(\mathrm{i}zc)}\right)_{z=z_j} - \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{xy}\frac{y(g(\mathrm{i}y))^2}{h(-yc)} \qquad x < 0. \tag{46}$$

In particular, one gets $f(x) \propto (\omega_0 x)^{-1}$ when $|x| \to \infty$.

The fact that $f(x) \neq 0$ for $x > 0$ is very unsatisfactory as it means that the field appears at a point of distance r from the system before the time $t = r/c$, which is in

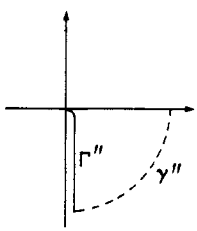

Figure 3. Contour of integration Γ''.

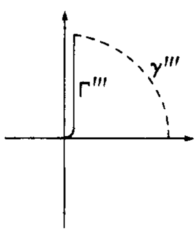

Figure 4. Contour of integration Γ'''.

contradiction with basic physical principles. There are several reasons which may cause this weakness in our treatment. Firstly, the description of our system by the hamiltonian (2) is non-relativistic. Secondly, we have used the dipole approximation and removed the field components with very large wavevectors $|\boldsymbol{k}|$. Thirdly, it may also happen that our initial conditions, ie the field in the ground state and the source in the excited state, are inconsistent with a relativistic description. These difficulties would not appear if the lower limit of the integral has been taken as $-\infty$. However, this procedure is not justified.

The contributions to the integrals (45) and (46) from the integration along the imaginary axis have no sinusoidal wave character. Only the poles of $(h(izc))^{-1}$ lying in the first quadrant contribute to a sinusoidal, damped wave train. Taking into account only the pole nearest to the real axis, one gets, for small coupling $\lambda \ll c$, the quasi-monochromatic part of the signal:

$$A_S^{\mathrm{rad}}(\boldsymbol{r}, t) = \mathrm{i}e\left(\frac{\hbar\omega_0 N}{2mc^2}\right)^{1/2} \exp\left[-\frac{1}{3}N\frac{e^2\omega_0^2}{mc^3}\left(t-\frac{r}{c}\right)\right]$$
$$\times\frac{1}{r}(\hat{I}-\boldsymbol{nn})\left\{\exp\left[\mathrm{i}\Omega\left(t-\frac{r}{c}\right)\right]\boldsymbol{b}^{\dagger}(0)-\exp\left[-\mathrm{i}\Omega\left(t-\frac{r}{c}\right)\right]\boldsymbol{b}(0)\right\} \tag{47}$$

where, cf (23),

$$\Omega = \omega_0\left(1-\frac{1}{3}\frac{e^2 N\omega_{\max}}{\pi mc^3}\right).$$

The angular distribution of the radiation is characteristic of dipole radiation.

The energy of the radiation corresponding to (47), neglecting terms of order $\mathrm{O}((\lambda/c)^4)$ is

$$E_S^{\mathrm{rad}} = \frac{1}{2\pi}\int \mathrm{d}\hat{\Omega}\int_0^{ct}\mathrm{d}r\, r^2 \boldsymbol{E}_S^{(+)\mathrm{rad}}(\boldsymbol{r}, t)\boldsymbol{E}_S^{(-)\mathrm{rad}}(\boldsymbol{r}, t) \simeq \hbar\omega_0\boldsymbol{b}^{\dagger}(0)\boldsymbol{b}(0) \tag{48}$$

where $\boldsymbol{E}^{(+)}$ and $\boldsymbol{E}^{(-)}$ denote those parts of the field which correspond to $\boldsymbol{b}^{\dagger}(0)$ and $\boldsymbol{b}(0)$, respectively.

The energy E_S^{rad} is thus equal to the total energy (40) of the field. This shows that the parts of the field which are related to the integrals along Γ'' and Γ''' do not carry energy, at least to our accuracy. This is some sort of justification for neglecting those parts of the signal which do not fulfil the causality condition.

6. Radiation in the presence of the A^2 term

Until now, we have been neglecting part of the interaction of $(e^2/2mc^2)\Sigma_{i=1}^{N}A^2$ type, which is present in the initial hamiltonian (1). In this section we are going to comment on the consequences of this term.

A correction to the hamiltonian (5) (11) coming from the $\boldsymbol{A}^2$ term, taken in the rotating wave and the dipole approximations, has the form:

$$H_{\mathrm{II}} = 2\hbar\frac{\lambda^2}{\omega_0}\sum_{\nu}\sum_{\mu}\int \mathrm{d}_3k\,\mathrm{d}_3p(\boldsymbol{e}_{\boldsymbol{k}\mu}\boldsymbol{e}_{\boldsymbol{p}\nu})g(k)g(p)a^{\dagger}_{\boldsymbol{k}\mu}a_{\boldsymbol{p}\nu}. \tag{49}$$

This term changes equation (13) for the operator $a^{\dagger}_{k\mu}(t)$ into:

$$\frac{\mathrm{d}}{\mathrm{d}t}a^{\dagger}_{k\mu} = \mathrm{i}\omega a^{\dagger}_{k\mu} + \lambda g(k)e_{k\mu}\,\boldsymbol{b}^{\dagger} + \mathrm{i}\frac{2\lambda^2}{\omega_0}g(k)e_{k\mu}\sum_{\nu}\int \mathrm{d}_3 p\, e_{p\nu}g(k)a^{\dagger}_{p\nu}. \tag{50}$$

The equation for $\boldsymbol{b}^{\dagger}(t)$ is the same as before. The system is still exactly solvable.

The solution remains in the form (14), (15) with functions $F(t)$, $G(k, t)$ and $J(k, p, t)$ modified according to the formulae:

$$F(t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{1-\mathrm{i}(2\lambda^2/\omega_0)\gamma(z)}{H(z)}\mathrm{e}^{zt}\,\mathrm{d}z \tag{51}$$

$$G(k, t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\mathrm{e}^{zt}}{(z-\mathrm{i}kc)H(z)}\mathrm{d}z \tag{52}$$

$$J(k, p, t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{1+\mathrm{i}2z/\omega_0}{(z-\mathrm{i}kc)(z-\mathrm{i}pc)H(z)}\mathrm{e}^{zt}\,\mathrm{d}z \tag{53}$$

where

$$\gamma(z) = \frac{8\pi}{3}\int_0^{\infty}\mathrm{d}k\frac{k^2(g(k))^2}{z-\mathrm{i}kc}$$

$$H(z) = z-\mathrm{i}\omega_0-\lambda^2\left(1+\frac{2\mathrm{i}z}{\omega_0}\right)\gamma(z).$$

As was mentioned in § 3, without the $\boldsymbol{A}^2$ term, the function $h(z)$ can have the zero on the negative part of the imaginary axis for sufficiently large coupling constant. Hence the bound state of the radiation and sources is possible.

One can easily verify that $H(z)$, the counterpart of $h(z)$, has no zero on the imaginary axis for any value of the coupling constant λ. This means that the bound state disappears.

In the case of weak coupling, $\lambda \ll c$, zeros of the function $H(z)$ are still given by (23), (24). This means that the $\boldsymbol{A}^2$ term does not affect the shape of the emitted line and the radiative damping constant in the lowest order of the coupling.

7. Final remarks

In our treatment we have neglected the motion of the oscillators, their collisions and electrostatic interactions. These processes are very important for the properties of radiation, if the emission is a slow process. However, as we have pointed out, due to the cooperation of the oscillators, the emission becomes very fast. For a system composed of a sufficiently large number of oscillators one can, in fact, disregard these slow processes resulting from motion and electrostatic interaction. They certainly have some influence, together with non-dipole radiation, in the later states of the evolution when, in our approximation, the system reaches the non-radiative state or, as is sometimes said, traps the radiation. The discussion given in this paper concerns the initial stage of the system evolution when the radiation is not disturbed by randomizing processes.

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Appendix

In this paper we use the integrals

$$I_\nu = \frac{8\pi}{3}\frac{\lambda^2}{c}\int_0^\infty \mathrm{d}\omega\omega^\nu \frac{k^2(g(k))^2}{|h(\mathrm{i}\omega)|^2} \qquad \nu = 0, 1. \tag{A.1}$$

Introducing the cut of the k plane along the positive part of the imaginary axis, we have the following relations:

$$\frac{8\pi}{3}\frac{\lambda^2}{c}k^2(g(k))^2 = \frac{1}{2\pi}(h(\mathrm{i}\omega+0)-h(\mathrm{i}\omega-0)) \tag{A.2}$$

$$|h(\mathrm{i}\omega)|^2 = h(\mathrm{i}\omega+0)h^*(\mathrm{i}\omega+0) = -h(\mathrm{i}\omega+0)h(\mathrm{i}\omega-0). \tag{A.3}$$

The integral (A.1) can be represented by the contour integral

$$I_\nu = \frac{1}{2\pi \mathrm{i}^{\nu+1}}\int_\gamma \mathrm{d}z\frac{z^\nu}{h(z)} \tag{A.4}$$

where the contour γ is shown in figure 5. With the cut as described above, the function $h(z)$ has no zeros in the z plane, cf (22), and this is the situation considered in this paper. Therefore, the integrand is an analytic function and the contour of integration γ can be deformed into γ', cf figure 5. The integral (A.4) can then be evaluated by expanding the integrand in powers of z^{-n}. In this way one finds

$$I_0 = 1, \qquad I_1 = \omega_0. \tag{A.5}$$

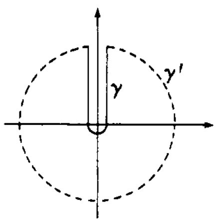

Figure 5. Contour of integration γ.

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